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Thrust Imparted to an Airfoil by Passage Through a Sinusoidal Upwash Field

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The passage of an airfoil through a sinusoidal upwash field (Sears, 1941) is analyzed further to yield the wavelength-dependent thrust: the "Katzmayr effect." The shed vorticity is shown to induce an opposed flow that reduces the kinetic energy: the decrement is found to match exactly the work done by the thrust. The "Sears function for thrust" is then applied to passage through turbulence described by a one-dimensional power spectrum. Finally, some acoustical implications of relaxing the postulated incompressibility are briefly discussed. It is argued that the radiated dipole noise, like the thrust, draws its energy from the flowfield.

Introduction

IN 1922, Katzmayr[1] reported on wind-tunnel experiments on airfoils for which "the direction of the air flow itself was subject to periodic oscillations." He found a region of small airfoil incidence showing negative drag, i.e., thrust. Similar effects for flight through random turbulence naturally suggest themselves. Phillips[2] has presented a simplified analysis of the approximate magnitude for some example cases for aircraft, allowing for the vertical response.

One can view the thrust as arising from a forward tilt of the lift vector in a region of upwash: it must be perpendicular to the inclined resultant flow vector. Likewise, in a region of downwash, the downward lift is again tilted forward. Thus, for a random or periodic up-down sequence, there is a nonzero average thrust.

Although his own experiments[1] prove the reality of the "Katzmayr effect," analytical evaluation—except via the aforementioned quasisteady argument[2]—seems to be lacking. The simplest scenario for a more general analysis is the passage of a thin flat airfoil through a sinusoidal upwash flowfield: two dimensional, incompressible, and inviscid. Sears[3] evaluated the lift response of the airfoil in terms of the much-cited "Sears function" of the frequency. Herein, we evaluate the associated thrust response.

But this is only a step—a major one—toward a larger objective: that is, to clarify the source of energy. The only possible pool is the kinetic energy of the upwash flowfield. Energy conservation would seem to require that the kinetic energy be reduced by precisely the work done by the thrust. That is to say, the wake of the airfoil must act to reduce the total kinetic energy. In more usual situations, the wake contribution is additive, so this seems counterintuitive. Thus, to explore this notion, we evaluate also the change in kinetic energy. The final step is then a comparison of the thrust work with the change in kinetic energy.

The physical concepts and associated analytical procedures draw primarily on Sears,[3] von Kármán and Sears,[4] von Kármán and Burgers,[5] and Garrick.[6] Sears's analysis for the lift did not, in itself, suffice: it did not evaluate the bound and trailing vortex sheets needed for determination of the thrust and kinetic energy.

The primary interest in the Katzmayr effect is for flight through turbulence. The present model can be applied to this more general scenario by Fourier methods. A formalism relating the mean thrust to the one-dimensional power spectrum of the upwash is developed. This is a generalization of a relation used by Phillips.[2]

The acoustical implications of relaxing the postulated incompressibility are discussed in a final section. An answer is offered to the long-standing conundrum of the origin of the radiated sound energy.

Analysis

The upwash field is modeled as arising by induction from a plane sinusoidal sheet of vorticity. This has the advantage that evaluation of the kinetic energy reduces to an integral along the sheet. The airfoil, of chord c, moves at zero incidence in the same plane with speed U toward the left (Fig. 1). In the figure we follow Ref. 4, and $\tilde{x}$ and $\tilde{\xi}$ referring to distance along the airfoil and wake, respectively. Reference 4 uses nondimensional distances, with $x = \tilde{x}/(c/2)$ and $\xi = \tilde{\xi}/(c/2)$; herein the tilde denotes dimensional values. All other quantities, except where noted, are dimensional.

The primary parameter is the nondimensional "reduced frequency,"

$$k = \tilde{k}c/2 = \pi cf/U = \pi c/\text{wavelength of upwash field} \quad (1)$$

where f is the frequency of the airfoil lift that results. The analysis is especially simple and physically transparent in the long wavelength limit $k \ll 1$. Although a wake vortex sheet is an outcome, its induction effects on the lift are neglected, being second order in k. This simple case is presented first, followed by the general analysis for arbitrary k.

Long Wavelength Limit ($k \ll 1$)

We use a frame of reference fixed in the fluid at rest far from the plane of the airfoil. In this frame, the airfoil is at $\tilde{x} = 0$ at $t = 0$. The prescribed sheet of vorticity is

$$\gamma_u(\tilde{\xi}) = 2\hat{w}_u \sin \tilde{k}\tilde{\xi} \quad (2)$$

which induces an upwash field in its plane[6]

$$w_u(\tilde{\xi}) = \hat{w}_u \cos \tilde{k}\tilde{\xi} \quad (3)$$

specified to be $\ll U$. For long wavelength $k \ll 1$, the angle of attack w_u/U will be essentially uniform along the chord of the moving airfoil. By classical quasisteady airfoil theory, the lift is

$$L = \rho U\Gamma = \pi\rho cUw_u \quad (4)$$

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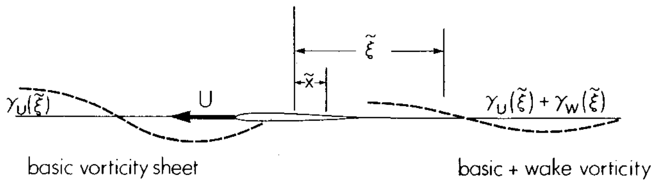

Fig. 1 Scenario and airfoil-attached coordinate system. Airfoil moves with speed U through sinusoidal upwash field induced by vorticity sheet $\gamma_u(\xi)$.

The airfoil circulation Γ will change by $d\Gamma$ in moving a distance $(-d\tilde{\xi})$ through the nonuniform w_u field. Conservation of circulation requires shedding vorticity $-\gamma_w \, d\tilde{\xi}$. It follows that

$$\gamma_w(\tilde{\xi}) = \frac{d\Gamma}{d\tilde{\xi}} \tag{5}$$

By Eqs. (3) and (4), this determines a wake vortex sheet as

$$\gamma_w(\tilde{\xi}) = -\pi c \tilde{k} \hat{w}_u \sin \tilde{k}\tilde{\xi} \tag{6}$$

In terms of the "reduced frequency" $k = \tilde{k}c/2$, the combined vortex sheet behind the airfoil is

$$\gamma(\tilde{\xi}) = \gamma_u + \gamma_w = 2(1 - \pi k)\hat{w}_u \sin \tilde{k}\tilde{\xi} \tag{7}$$

which induces an upwash field

$$w(\tilde{\xi}) = w_u + w_w = (1 - \pi k)\hat{w}_u \cos \tilde{k}\tilde{\xi} \tag{8}$$

in its plane (part of a spatial flow).

The kinetic energy of this flow between two infinite normal planes is (justified later)

$$KE = -\frac{1}{2}\int_a^{a+\mathcal{L}} w(\tilde{\xi})\left[\int \gamma(\tilde{\xi}')\, d\tilde{\xi}'\right] d\tilde{\xi} \tag{9}$$

where the separation $\mathcal{L}$ is taken to be an integral number of half wavelengths. Upon evaluation, using Eq. (1) for $\tilde{k}$,

$$KE = \left(\frac{\rho c \hat{w}_u^2}{2k}\right)(1 - \pi k)^2 \int_a^{a+\mathcal{L}} \cos^2 \tilde{k}\tilde{\xi} \, d\tilde{\xi} \tag{10}$$

$$= \frac{\rho c \hat{w}_u^2 \mathcal{L}}{4k}(1 - 2\pi k + \pi^2 k^2) \tag{11}$$

The incremental kinetic energy imparted by the wake is thus

$$\Delta KE = \frac{\rho c \hat{w}_u^2 \mathcal{L}}{4k}(-2\pi k + \pi^2 k^2) \tag{12}$$

the dimensional factor being just the basic upwash kinetic energy. Next this is to be compared with the thrust work in moving through a distance $\mathcal{L}$. The relative flow is inclined to the horizontal by angle $\alpha = w_u/U$. The lift L, being normal to the flow vector, is likewise tilted. The thrust is the forward component

$$T = (w_u/U)L = \pi \rho c w_u^2 \tag{13}$$

by Eq. (4). (Physically, this is attributed to suction along the leading edge of the airfoil.) The average thrust over an integral number of half wavelengths is

$$\langle T \rangle = \pi \rho c \hat{w}_u^2 / 2 \tag{14}$$

using Eq. (3) for w_u. The work done in moving a distance $\mathcal{L}$ is

$$\langle T \rangle \mathcal{L} = \frac{\rho c \hat{w}_u^2 \mathcal{L}}{4k}(2\pi k) \tag{15}$$

in a form comparable to that of Eq. (12). It seems that the ( ) factors in Eqs. (12) and (15) agree to the first order in πk, but with opposite sign: *the thrust work $\langle T \rangle \mathcal{L}$ is paid for by a loss in fluid kinetic energy ΔKE.* (The second-order difference is spurious, arising in part from a neglect in wake induction effects: see later.)

General Case (Any k)

For this analysis, we switch to a reference frame fixed in the airfoil of Fig. 1. In this frame, the vortex sheets are convected in the positive $\tilde{\xi}$ direction with speed $U = \nu/\tilde{k}$. The prescribed vortex sheet and its induced upwash field, Eqs. (1) and (2), go over to

$$\gamma_u = 2i\hat{w}_u \exp[i(\nu t - \tilde{k}\tilde{\xi})] \tag{16}$$

$$w_u = \hat{w}_u \exp[i(\nu t - \tilde{k}\tilde{\xi})] \tag{17}$$

respectively, in complex form.

The lift response of the airfoil consists of three components[4]: 1) a quasisteady part

$$\rho U \Gamma_0 = \rho U \int_{-c/2}^{c/2} \gamma_0(\tilde{x})\, d\tilde{x} \tag{18}$$

derived from the upwash field w_u, Eq. (16), without regard for its unsteadiness; 2) a part associated with bound vorticity $\gamma_1(\tilde{x})$ induced by the upwash field of the wake $\gamma_w(\tilde{\xi})$; and 3) an inertial part associated with fluid acceleration (virtual mass effect). Herein, we must obtain $\gamma_0(\tilde{x})$ and $\gamma_1(\tilde{x})$ near the leading edge to determine the thrust, and we must determine the shed vorticity field $\gamma_w(\tilde{\xi})$ for evaluation of the wake-associated change in fluid kinetic energy. The interrelation is such that the sequence of derivation is $\gamma_0(\tilde{x})$, $\gamma_w(\tilde{\xi})$, and $\gamma_1(\tilde{x})$.

Quasisteady Airfoil Vorticity $\gamma_0(\tilde{x})$

Von Kármán and Burgers (Ref. 5, p. 38) consider, in effect, an airfoil in a steady upwash field of form

$$w_u(\theta) = A_0 + \sum_1^\infty A_m \sin m\theta / \sin\theta; \qquad \tilde{x} = \frac{c}{2}\cos\theta \tag{19}$$

and determine the induced bound vorticity $\gamma_0(\theta)$. But by trigonometric sum and difference formulas, this can be manipulated into the form[7]

$$w(\theta) = B_0 + 2\sum_1^\infty B_n \cos n\theta \tag{20}$$

inducing the vorticity

$$\gamma_0(\theta) = 2B_0 \frac{1 - \cos\theta}{\sin\theta} + 2\sum_1^\infty B_n \sin n\theta \tag{21}$$

along the airfoil. The corresponding circulation, normalized by $c/2$, is

$$\Gamma_0 = \frac{2}{c}\int_{-c/2}^{c/2} \gamma_0(\theta)\, d\tilde{x} = 2\pi(B_0 + B_1) \tag{22}$$

In our application this is quasisteady, the time dependence being allowed for in B_0 and B_1. The Γ_0 takes the form

$$\Gamma_0 = 2\pi(B_0' e^{i\nu t} + B_1' e^{i\nu t}) \equiv G_0 e^{i\nu t} \tag{23}$$

Wake Vorticity $\gamma_w(\tilde{\xi})$

As the solution of an integral equation, Ref. 4, Eqs. (21-25) relate the wake vorticity to an arbitrary quasisteady circulation $G_0 e^{i\nu t}$; this yields

$$\gamma_w(\tilde{\xi}) = -\frac{G_0}{K_0(ik) + K_1(ik)} \exp[i(\nu t - \tilde{k}\tilde{\xi})]$$

$$= -\frac{2\pi(B_0' + B_1')}{K_0(ik) + K_1(ik)} \exp[i(\nu t - \tilde{k}\tilde{\xi})] \tag{24}$$

where the second equality reflects the choice of G_0 specified in Eq. (23). Here $K_0(ik)$ and $K_1(ik)$ are modified Bessel functions of the second kind and argument ik, k being the reduced frequency [Eq. (1)]. Sears[3] applied the expansion

$$\exp\left[-ik\cos\theta\right] = J_0(k) + 2\sum_{1}^{\infty}(-i)^n J_n(k)\cos n\theta \tag{25}$$

with $\tilde{\xi} = (c/2)\cos\theta$, to the $e^{-i\tilde{k}\tilde{\xi}}$ factor of Eq. (16) to throw it into the form of Eq. (20); the $J_n(k)$ are Bessel functions of the first kind. Comparison serves to evaluate the B_n of Eq. (21) and the corresponding B_0' and B_1'. Insertion in Eq. (24) yields the wake vorticity as

$$\gamma_w(\tilde{\xi}) = -2\pi\,\frac{J_0(k) - iJ_1(k)}{K_0(ik) + K_1(ik)}\,\hat{w}_u \exp\left[i(\nu t - \tilde{k}\tilde{\xi})\right] \tag{26}$$

Wake-Induced Airfoil Vorticity $\gamma_1(\tilde{x})$

Equation (7) of Ref. 4 gives the vorticity induced along the airfoil by a vortex of circulation Γ' at point $\tilde{\xi}$ in the wake. Inserting $\gamma_w(\tilde{\xi})\,\mathrm{d}\tilde{\xi}$ for Γ' and integrating yields the total vorticity induced by the wake as

$$\gamma_1(\tilde{x}) = \frac{1}{\pi}\sqrt{\frac{c/2 - \tilde{x}}{c/2 + \tilde{x}}}\int_{c/2}^{\infty}\frac{1}{\tilde{\xi} - \tilde{x}}\sqrt{\frac{\tilde{\xi} + c/2}{\tilde{\xi} - c/2}}\,\gamma_w(\tilde{\xi})\,\mathrm{d}\tilde{\xi} \tag{27}$$

in our dimensional notation. For evaluation of the suction thrust, only the limiting form near the leading edge $\tilde{x} = -c/2$ is required. The term $\tilde{\xi} - \tilde{x}$ may be approximated there as $\tilde{\xi} + c/2$, leading, after insertion of Eq. (26) for $\gamma_w(\tilde{\xi})$, to

$$\gamma_1(\tilde{x}) = -2\hat{w}_u e^{i\nu t}\,\frac{J_0 - iJ_1}{K_0 + K_1}\sqrt{\frac{c/2 - \tilde{x}}{c/2 + \tilde{x}}}\int_{c/2}^{\infty}\frac{e^{-i\tilde{k}\tilde{\xi}}}{\sqrt{\tilde{\xi}^2 - c^2/4}}\,\mathrm{d}\tilde{\xi} \tag{28}$$

where the arguments of the various J and K have been omitted. The integral is evaluated in Ref. 4 in the equation after Eq. (24), as $K_0(ik)$. Further, with s equal to $\tilde{x} + c/2$, the distance from the leading edge, the square root factor is just $\sqrt{c/s}$. Thus

$$\gamma_1(\tilde{x}) \xrightarrow[\tilde{x}\to -c/2]{} -2\hat{w}_u e^{i\nu t}\,\frac{K_0(J_0 - iJ_1)}{K_0 + K_1}\sqrt{\frac{c}{s}} \tag{29}$$

Leading-Edge Singularity

The suction thrust depends on both $\gamma_1(\tilde{x})$ and $\gamma_0(\tilde{x})$ near the leading edge. In that vicinity, only the $(1 - \cos\theta)/\sin\theta$ term of γ_0, Eq. (21), is significant, and for $\tilde{x} \to -c/2$ this likewise approaches the singularity $\sqrt{c/s}$. Thus with B_0 equal to $\hat{w}_u J_0 e^{i\nu t}$ [implicit in Eqs. (22–26)],

$$\gamma_0(\tilde{x}) \xrightarrow[\tilde{x}\to -c/2]{} 2\hat{w}_u e^{i\nu t} J_0(k)\sqrt{c/s} \tag{30}$$

The combined singularity in vorticity at the leading edge is the sum of $\gamma_0(\tilde{x}) + \gamma_1(\tilde{x})$. We write this as

$$\gamma(\tilde{x}) = 2\hat{w}_u S(k) e^{i\nu t}\sqrt{c/s} \tag{31}$$

$$S(k) \equiv J_0(k) - \frac{K_0(ik)[J_0(k) - iJ_1(k)]}{K_0(ik) + K_1(ik)} = \frac{J_0(k)K_1(ik) + iJ_1(k)K_0(ik)}{K_0(ik) + K_1(ik)} \tag{32}$$

(See also Amiet.[8]) The function $S(k)$ can be recognized as the Sears function for the lift response.[3]

Suction Thrust

The singularity $\sqrt{c/s}$ in the vorticity implies an infinite velocity around the leading edge of a mathematically thin plate: an edge of zero radius of curvature. This may be regarded as an idealization of a high finite velocity around a rounded edge of a plate of small thickness. By Bernoulli's equation, there is a forward suction force—a thrust—per unit length of edge. If the vorticity is written in the form

$$\gamma = A\sqrt{c/s} \tag{33}$$

then, invoking only the real part,[6] the leading-edge suction thrust can be obtained by a limiting process as

$$T = (\pi/4)\rho c A_r^2 \tag{34}$$

(cf. also Ref. 5, p. 52, and earlier references cited therein). Putting ϵ as the phase angle of the complex $S(k)$ in Eq. (31), we take

$$A_r = 2R_e \hat{w}_u\,|S|\,e^{i\epsilon}e^{i\nu t} \tag{35}$$

$$= 2\hat{w}_u\,|S|\cos(\nu t + \epsilon) \tag{36}$$

The instantaneous suction thrust is then

$$T = \pi\rho c\hat{w}_u^2\,|S|^2\cos^2(\nu t + \epsilon) \tag{37}$$

$$= (\pi/2)\rho c\hat{w}_u^2\,|S|^2[1 + \cos 2(\nu t + \epsilon)] \tag{38}$$

It is evident that the thrust oscillates about a mean at twice the frequency of the upwash w: it peaks on both the upflow and the downflow.

The mean thrust, normalized by its peak value, is just

$$\langle T\rangle/\langle T(0)\rangle = |S(k)|^2 \tag{39}$$

The variation with k ($\propto$ chord/wavelength) is shown in Fig. 2 (prepared with data for S from Ref. 9).

Sears Function for Thrust

The real part of the oscillatory lift response of the airfoil derived by Sears[3] is

$$L = (\pi\rho cU\hat{w})R_e[S(k)e^{i\nu t}] \tag{40}$$

wherein $S(k)$ is referred to as the Sears function for the lift.

For comparison, the thrust Eq. (38) may be written in the alternative form

$$T = (\pi\rho c\hat{w}^2/2)\{\,|S(k)|^2 + R_e[S^2(k)e^{i2\nu t}]\} \tag{41}$$

Here the factor $|S(k)|^2$ accounts for the mean thrust (there is no mean lift) and $Re[S^2(k)e^{i2\nu t}]$ for the periodic part (of double the lift frequency). Thus, with a caveat for these differences, we suggest that $S^2(k)$ might be designated the "Sears function for the thrust." (Note that deletion of R_e from Eq. (41) will not result in a valid complex equation: only the real part is physically applicable. The complex Sears equation, of which Eq. (40) is the real part, is not so restricted.)

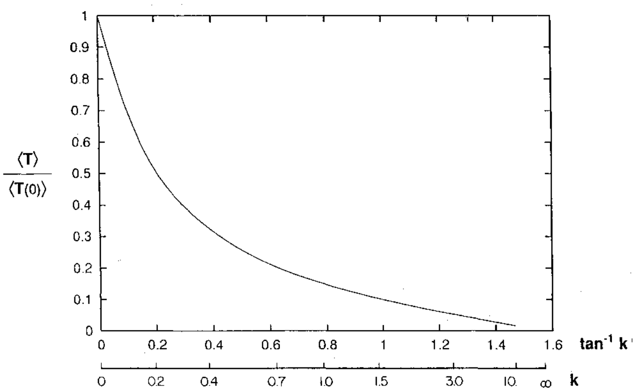

Fig. 2 Normalized time-average thrust ⟨T⟩ vs "reduced frequency" k (= π chord/wavelength). This is a kind of "Sears function" for thrust.

Incremental Kinetic Energy

We wish to compare the thrust work with the incremental kinetic energy of the upwash field imparted by the wake vorticity. This is reckoned relative to a fluid-fixed reference frame, with the airfoil advancing to the left (Fig. 1). The airfoil velocity U makes no contribution to the energy, but the airfoil-fixed reference frame of Fig. 1, treating U as a flow velocity, is retained for convenience.

The kinetic energy induced by a vortex sheet between two infinite normal planes is

$$KE = -\tfrac{1}{2}\rho\int_{\xi_0}^{\xi_0+\mathcal{L}} (\varphi_2-\varphi_1)w\,d\bar{\xi} \tag{42}$$

where the separation $\mathcal{L}$ is an integral number of half wavelengths. Counting both sides, this is the form taken by $-\tfrac{1}{2}\int\rho\phi\partial\varphi/\partial n\,dS$ (Ref. 6). Here $(\varphi_2-\varphi_1)$ is the potential jump from the top to the bottom of the sheet, given by the indefinite integral

$$\varphi_2-\varphi_1 = \int\gamma\,d\bar{\xi} \tag{43}$$

As for the thrust, these equations are restricted herein to real quantities. Thus, in what follows, we refer where appropriate to the real part of the complex quantities derived earlier. The combined primary and wake vortex sheets are given by the sum of Eqs. (16) and (26). Using $i = e^{i\pi/2}$

$$\gamma = \gamma_u + \gamma_w = 2\hat{w}_u \exp\left[i\left(\frac{\nu t - \bar{k}\bar{\xi} + \pi}{2}\right)\right]$$

$$\times\left[1 + i\pi\frac{J_0(k) - iJ_1(k)}{K_0(ik)+K_1(ik)}\right] \tag{44}$$

Abbreviating the term in brackets,

$$G = |G|e^{ig} = 1 + i\pi\frac{J_0(k)-iJ_1(k)}{K_0(ik)+K_1(ik)} \tag{45}$$

so that the real part of γ is

$$\gamma_r = 2\hat{w}_u|G|\cos\left(\nu t - \bar{k}\bar{\xi} + \frac{\pi}{2} + g\right) \tag{46}$$

$$= -2\hat{w}_u|G|\sin(\nu t - \bar{k}\bar{\xi} + g) \tag{47}$$

By Eq. (43), the integral over $\bar{\xi}$ yields

$$\varphi_2-\varphi_1 = -\frac{2}{\bar{k}}\hat{w}_u|G|\cos(\nu t - \bar{k}\bar{\xi} + g) \tag{48}$$

The upwash field in the plane of the airfoil is inferred from the complex form of the vorticity field γ by division by $2i = 2e^{i\pi/2}$ [cf. Eqs. (16) and (17)]. Thus, from the previous equations, the real part is

$$Re\,w = \hat{w}_u|G|\cos(\nu t - \bar{k}\bar{\xi} + g) \tag{49}$$

Inserting in Eq. (42), replacing $\bar{k}$ by $2k/c$,

$$KE = \frac{\rho c}{2k}\hat{w}_u^2|G|^2\int_a^{a+\mathcal{L}}\cos^2(\nu t - \bar{k}\bar{\xi} + g)\,d\bar{\xi} \tag{50}$$

$$= (\rho c/4k)\hat{w}_u^2|G|^2\mathcal{L} \tag{51}$$

This may be split into a basic part KE_0 associated with the upwash field w_u and an increment ΔKE owing to passage of the airfoil:

$$KE_0 = (\rho c/4k)\hat{w}_u^2\mathcal{L} \tag{52}$$

$$\Delta KE = (\rho c/4k)\hat{w}_u^2(|G|^2-1)\mathcal{L} \tag{53}$$

After considerable reduction (see the Appendix) $(|G|^2-1)$ may be re-expressed as just $-2\pi k|S(k)|^2$, S being just the Sears function, Eq. (32) [or alternatively, Eq. (A2)]. Thus, by virtue of Eq. (52),

$$\Delta KE = -2\pi k KE_0|S(k)|^2 \tag{54}$$

What Powers the Thrust?

To answer this, we compare the thrust work on traversing a distance $\mathcal{L}$ with the corresponding incremental kinetic energy. The work done is $\mathcal{L}$ times the average value of T, given by Eq. (38) without the cosine term. On normalizing by $2\pi k KE_0$, the result is

$$\frac{\text{thrust work}}{2\pi k KE_0} = |S(k)|^2 = (S_r^2 + S_i^2) \tag{55}$$

where KE_0 is the original kinetic energy of the fluid traversed by the airfoil (see earlier). The wake shed in the process induces an incremental kinetic energy ΔKE given by Eq. (54). When similarly normalized,

$$\frac{\Delta KE}{2\pi k KE_0} = -|S(k)|^2 = -(S_r^2 + S_i^2) \tag{56}$$

Here again, *the thrust work is seen to be offset exactly by a kinetic energy decrement.* This generalizes the long wavelength $(k\to 0)$ conclusion [f. Eq. (15)] to apply to all wavelengths (any k).

Extension to Flight Through Turbulence

For homogeneous turbulence, the mean square upwash in the plane of flight may be written as a two-dimensional power spectrum,[10] in nondimensional form,

$$\bar{w}^2 = \int_\infty\int_\infty \Psi_2(k_1, k_2)\,dk_2\,dk_1 \tag{57}$$

where $k_1 = \pi c/\lambda_1$ and $k_2 = \pi c/\lambda_2$ refer to wavelengths along the airfoil chord and span, respectively. Spanwise variations are integrated out (ignored) in the one-dimensional spectrum

$$\bar{w}^2 = \int_\infty \Psi_1(k_1)\,dk_1 = \int_\infty \Psi(k)\,dk \tag{58}$$

in which Ψ_1 is the inner integral in Eq. (57) and we drop the subscript 1. For a corresponding power spectrum of $\langle A^2\rangle$ governing the thrust [Eqs. (33), (34)] we need the transfer function relating the (complex) amplitude of A to $\hat{w}_u$: comparison of Eqs. (31) and (32) gives

$$A = \hat{A}e^{i\nu t} = 2S(k)\hat{w}_u e^{i\nu t} \tag{59}$$

in the airfoil-attached reference frame of Fig. 1, and

$$A = \hat{A}e^{-ikx} = 2S(k)\hat{w}_u e^{-ikx} \qquad (kx = \bar{k}\bar{x}) \tag{60}$$

in the fluid-attached reference frame of Ref. 10 (with different sign convention). Thus $2S(k)$ is the desired transfer function. For application to power spectra we reinterpret $\hat{A}$ and $\hat{w}$ as differentials relating to a (nondimensional) wave number band dk centered on k. Then by Eq. (18) of Ref. 10

$$\langle A^2\rangle = 4\int_\infty \Psi(k)|S(k)|^2\hat{w}_u^2\,dk \tag{61}$$

A corresponding relation for the real part of A is

$$\langle A_r^2\rangle = 2\int\Psi(k)|S(k)|^2\hat{w}_u^2\,dk \tag{62}$$

so that from Eq. (34), the time-average thrust is

$$\langle T\rangle = (\pi/2)\rho c\hat{w}_u^2\int_\infty\Psi(k)|S(k)|^2\,dk \tag{63}$$

Phillips [Ref. 2, Eq. (3)] obtains an equivalent integral for an aircraft. The factor corresponding to $|S(k)|^2$ in the integrand is specialized to the infinite wavelength ($k = 0$) limit, and the decay at finite wavelengths is replaced (in a heuristic estimate) by an upper k cutoff. Our results, Eq. (63), apply to an *airfoil* constrained to a rectilinear flight path, whereas Phillips's result applies to an *aircraft* not so constrained; it therefore allows for the dynamical vertical response that progressively weakens the thrust (and lift) as $k \to 0$ ($\lambda \to \infty$). Some of his example computations are discussed in the next section.

Equation (63) does not take account of spanwise variations in upwash [cf. Eqs. (57) and (58)]. For these components of the turbulence, the one-dimensional transfer function of Eqs. (59–62) is no longer an adequate approximation to the airfoil response. For the lift of a spanwise strip, the appropriate two-dimensional transfer function has been derived by Filotas[11] in a generalization of Sears's problem (see also Refs. 12–15, among others). And in Ref. 16 he has incorporated this two-dimensional transfer function into a formalism for airfoil lift response to two-dimensional turbulence. A further very clever development[17] extends these results to apply, with some approximation, to finite wings of high aspect ratio. It would appear that such methods could be adapted to deal with the thrust problem in these more general scenarios.

Discussion

In flight through turbulence, the thrust from the Katzmayr effect will be very small compared with the lift. Phillips[2] estimated the thrust for several aircraft: a soaring glider, a light airplane, a fighter, and a transport. Typical sizes and weights were assumed, with a turbulence scale of 300 m and $C_L = 1.0$. In all cases the turbulence scale ($\propto$ dominant wavelength) is large compared with the chord. Thus, for our sinusoidal scenario all cases are close to the long wavelength limit ($k = 0$): the predicted decay in thrust with increasing k does not come into play. On the other hand, the aircraft vertical dynamic response, which Phillips takes into account, does reduce the long wavelength contribution. He comes up with an estimate of thrust for a glider ranging from 10 to 20% of the drag; this seems to be an upper limit. Thus, although Katzmayr was able to measure the effect in his stronger experimental periodic flow, it is of doubtful observability in turbulence.

A variation on the Katzmayr effect is the added thrust provided by stators behind rotors. This is in purest form when the stators are at zero incidence to the main stream: then the swirl provides the entire angle of attack. The energetics here are much more transparent: the swirl and its kinetic energy are clearly reduced to power the thrust.

Acoustical Implications

If we relax the postulated incompressibility, the effect has acoustical implications as well. It is well known[18] that a flat plate in turbulent flow will radiate noise. The dominant sound is, in effect, emitted by surface dipoles of strength measured by the lifting pressure: the predicted correlation of the fluctuating lift with the radiated sound has been confirmed experimentally.[19] Clearly, there should be a secondary emission from leading-edge dipoles. Because it scales with the thrust, this should be very much weaker than the lift emission.

The figure-of-eight dipole directional pattern of the thrust noise will be directed fore and aft rather than normal to the airfoil as is the lift noise pattern. (If the airfoil is in a shear flow, e.g., a jet, the thrust noise pattern will be markedly distorted by refraction, and the lift noise pattern will be hardly altered.) Further, the frequency doubling found for thrust in the periodic scenario implies a spectral shift in the case of turbulence: the thrust noise spectrum should resemble the lift noise spectrum, but upshifted an octave.

The energy source for the fluctuating lift noise has been controversial. (The second-order thrust noise was not noticed.) Clearly, the mechanism is different from the radiation of sound by a vibrating plate: in that case the sound energy is readily traced to the work done by the plate. Now suppose we generalize the present scenario to allow for compressibility. Once again we would infer that the flowfield energy (in this case kinetic plus internal) would power the Katzmayr thrust. But now it must power the radiated sound field as well: there is no other store of energy. This argument, supported by the proof herein concerning the thrust power in the incompressible case, would appear to answer the lift noise conundrum.

Lighthill[20] has suggested that work by the airfoil against "wave drag" could supply the energy of the lift noise. It was, in fact, exploration of this idea that led to the present author's rediscovery of the Katzmayr thrust. This arose from the realization that only a suction force could be supported by the leading edge of a flat plate airfoil. However, the compressible Katzmayr thrust could be reinterpreted as a basic thrust minus a wave drag. It is in this sense that Lighthill's concept could be applied. But, more fundamentally, all of the energy—that to supply the work of the net thrust and that to supply the acoustic wave field—must come from the flowfield through which the airfoil moves.

Appendix: Evaluation/Reduction of $|G|^2 - 1$ in Eq. (53)

By definition, Eq. (45),

$$G(k) = 1 + i\pi \frac{J_0(k) - iJ_1(k)}{K_0(ik) + K_1(ik)} \tag{A1}$$

Now consider the Sears function $S(k)$, Eq. (32): a simpler form due to Kemp[9] and tabulated by him is

$$S(k) = 1/ik[K_0(ik) + K_1(ik)] = S_r + iS_i \tag{A2}$$

Although the K_0 and K_1 will reappear, it will be algebraically convenient to use Eq. (A2) to eliminate them temporarily from G in terms of S:

$$G(k) = 1 - \pi k S(k)[J_0(k) - iJ_1(k)] \tag{A3}$$

Then, on expansion,

$$|G|^2 - 1 = \pi^2 k^2 (J_0^2 + J_1^2)(S_r^2 + S_i^2) - 2\pi k (S_r J_0 + S_i J_1) \tag{A4}$$

To simplify further, we invoke the following relations cited by Kemp[9]:

$$K_0(ik) = -(\pi/2)i[J_0(k) - iY_0(k)] \tag{A5}$$

$$K_1(ik) = -(\pi/2)[J_1(k) - iY_1(k)] \tag{A6}$$

It follows that an alternative form of S is

$$S(k) = 2/\pi k E(k); \qquad |S|^2 = (2/\pi k)^2/|E|^2 \tag{A7}$$

where

$$E = (J_0 - Y_1) - i(J_1 + Y_0) \tag{A8}$$

so that

$$S_r = (2/\pi k)(J_0 - Y_1)/|E|^2; \quad S_i = (2/\pi k)(J_1 + Y_0)/|E|^2 \tag{A9}$$

The first term of Eq. (A4) is thus

$$\pi^2 k^2 (J_0^2 + J_1^2)(S_r^2 + S_i^2) = 4(J_0^2 + J_1^2)/|E|^2 \tag{A10}$$

and the second term is

$$-2\pi k (S_r J_0 + S_i J_1) = -4(J_0^2 + J_1^2 + J_1 Y_0 - J_0 Y_1)/|E|^2 \tag{A11}$$

By a well-known relation among Bessel functions, again quoted in Ref. 9,

$$J_1 Y_0 - J_0 Y_1 = 2/\pi k \tag{A12}$$

On combining the three relations, Eq. (A4) reduces to

$$|G|^2 - 1 = -8/\pi k |E|^2 \tag{A13}$$

and by the further step, Eq. (A7),

$$|G|^2 - 1 = -2\pi k |S|^2 = -2\pi k(S_r^2 + S_i^2) \tag{A14}$$

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